Colorings, cliques, and independent sets in graph classes

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(abstract)

PI: Bartosz Walczak

Jagiellonian University in Kraków

The project aims at solving open problems on combinatorial and algorithmic properties of colorings, cliques, and independent sets in classes of graphs defined in two standard ways:

(a) graphs with geometric representations, such as intersection/disjointness graphs of specific geometric objects (rectangles, segments, curves, etc.), where the objects become the vertices and the pairs of vertices that intersect/are disjoint become the edges,

(b) graphs excluding a fixed structure H as an induced subgraph (so-called H-free graphs), with particular focus on the case $H = P_t$ (a path on t vertices).

On the combinatorial side, the project focuses on the interplay of three fundamental graph parameters: chromatic number (the minimum number of colors in a proper vertex coloring), clique number (the maximum size of a clique), and independence number (the maximum size of an independent set). A class of graphs is *χ-bounded* if the chromatic number of graphs in that class is at most some function of the clique number, and it is *polynomially χ-bounded* if that function is polynomial. A class of graphs has the *Erdős-Hajnal property* if there is a constant c such that every n-vertex graph in the class contains a clique or an independent set of size at least n^c. Polynomial χ-boundedness implies the Erdős-Hajnal property. Recent years have brought very significant progress of understanding which classes of graphs are χ-bounded. The present project aims at

(1) determining the order of the magnitude of the χ-bounding functions for the classes of rectangle graphs, complements of rectangle graphs, and circle graphs, which are known to be polynomially χ-bounded;

(2) deciding whether known χ-bounded classes of graphs, such as P_t-free graphs (with $t \geqslant 5$) and outer-string graphs (intersection graphs of curves touching a fixed line), are polynomially χ-bounded;

(3) deciding whether the classes of P_t-free graphs (with $t \geqslant 5$) and classes related to them satisfy the Erdős-Hajnal property;

(4) improving the lower or upper bounds on the chromatic number in terms of the number of vertices, when the clique number is bounded, for classes of geometric intersection or disjointness graphs that are not χ-bounded (L-graphs, segment graphs, string graphs, co-string graphs, co-outer-string graphs).

Some graphs with geometric representations, such as outer-string graphs or polygon visibility graphs, come with a natural ordering of the vertices and, consequently, can be described in terms of excluded induced ordered subgraphs. Therefore, another objective of the project is to decide for which ordered graphs H the class of H-free ordered graphs is (polynomially) χ-bounded. On the algorithmic side, the project focuses on exact and approximation algorithms for the maximum independent set problem (MIS). One direction of research is understanding the complexity of this problem for H-free graphs; for example, for $H = P_t$ with $t \geqslant 7$, neither a polynomial-time algorithm nor an NP-hardness proof are known. For the open cases (which also include H being subdivisions of a claw), the main objective is to find new polynomial-time or subexponential-time algorithms. For most classes of geometric intersection graphs (rectangle graphs, disk graphs, etc.), the MIS problem is NP-complete and research is primarily focused on approximation algorithms. Here, the objective is to improve the best known approximation factors, such as $O(\log \log n)$ for rectangle graphs and $O(n^a)$ for string graphs. We also plan to investigate whether conditional lower bounds (NP-hardness, ETH-hardness, etc.) can be proved for MIS on selected graph classes.