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Parity and minors in graph sharing games

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1 Introduction

Graph sharing games are played on a simple connected graph with non-negative weights placed on the vertices. Two players make their moves consecutively by pointing vertices one by one and removing them from the graph. The vertices available to play depend on the game variant:

- *Game T (Taken part connected)*: after each move the taken part of the graph should stay connected.
- *Game R (Remaining part connected)*: after each move the remaining part of the graph should stay connected.

The game ends when there are no more vertices. Both players' goal is to maximize the total weight they have collected at the end.

The two variants of the game coincide when the game is played on a cycle. This case was studied by Brown, who conjectured that the first player can always secure $\frac{1}{2}$ of the weight on any cycle. This is true for cycles with an even number of vertices but occurred to be false in general. Later Winkler conjectured that the first player can secure at least $\frac{4}{9}$ of the weight of any cycle, and this has been proved by two independent groups of researchers [2, 1].

After solving the problem for cycles it was natural to ask what happens for general graphs. Game T and game R are two ways to generalise the game to a broader family of graphs and both have been studied. In both games there are trees on which the first player can capture only one non-zero vertex and the second player can take arbitrarily many, but this can happen only for odd trees in game R and for even trees in game T. Micek and Walczak [5, 6] showed that the first player can capture $\frac{1}{4}$ of the total weight in game T on odd trees and in game R on even trees. Later Seacrest and Seacrest [8] improved the result for game R: the first player can capture $\frac{1}{2}$ of the weight on even trees.

Parity plays an important role but is not enough to explain the behaviour of the game in the general case. There are examples, which will be explained in Chapter 3, showing that the first player can be left with just one non-zero vertex on general graphs with both parities. In 2008 Micek and Walczak [6] stated the following conjecture, which is the main topic of this thesis.

Conjecture 1.1. *There is a function $f(n) > 0$ such that the first player can secure at least $f(n)$ of the weight of any graph with an odd number of vertices and with no K_n -minor in game T.*

We will prove the following, weaker statement:

Theorem 1.2. *If G is a $\{0,1\}$ -weighted graph with an odd number of vertices and $K_n \not\leq G$ then the first player can secure at least $\Omega(\frac{1}{(n\sqrt{\log n})^3})$ of the total weight of G in game T.*

2 Basics: definitions and tools

A *graph* is a pair $G = (V, E)$ where V is a set of *vertices* and $E \subseteq \binom{V}{2}$ is a set of *edges*. Given a graph G we denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G . All graphs considered have no loops (edges connecting a vertex to itself). Two vertices are *adjacent* if there is an edge between them. A *path* in a graph G is a sequence $v_0, \dots, v_k$ of vertices of G such that v_i is adjacent to v_{i+1} for every i . We call k the *length* of the path $v_0, \dots, v_k$. A *cycle* is a path in which $v_0 = v_k$. A *clique* K_n is a graph with n vertices such that every two vertices are adjacent. The *distance* between two vertices is the length of a shortest path connecting them. The *neighbourhood* of a set $B \subseteq V(G)$ is the set of all vertices from $V(G) \setminus B$ adjacent to any vertex in B . We denote it by $N_G(B) = \{v \in V(G) \setminus B : \exists b \in B \text{ } v \text{ and } b \text{ are adjacent}\}$. We define $N_G^+(B) = N_G(B) \cup B$. We write simply $N(B)$ and $N^+(B)$ when the graph G is clear from the context. A *multigraph* is a pair (V, E) where V is a set of vertices and E is a multiset of edges (that is, two vertices can share more than one edge). If $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ then G' is a subgraph of G . If $V(G') \subseteq V(G)$ and $E(G')$ contains all edges $xy \in E$ with $x, y \in V(G')$ then G' is an *induced subgraph* of G . For a set $A \subseteq V(G)$ by $G[A]$ we denote the induced subgraph of G whose set of vertices is A . A graph G is *connected* if there is a path connecting any two vertices of G .

A *vertex colouring* of a graph G is a map $c : V(G) \rightarrow S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. Elements of S are called colours. If $|S| = k$ then c is called a *k-colouring*. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is *k-colourable*.

Let H be a connected subgraph of G . By G/H we denote the graph obtained from G by *contracting* H into a single vertex v_H , that is, by replacing H by a vertex v_H which becomes adjacent to all neighbours of any vertex from H . A graph G' is a *minor* of G (denoted by $G' \prec G$) if G' can be obtained from G as a result of a sequence of the following *basic contractions*:

- deleting an edge,
- deleting a vertex,
- subgraph contraction.

There is an equivalent definition of a minor using a notion of touching sets. Two sets $A, B \subseteq V(G)$ are *touching* if $N^+(A) \cap B \neq \emptyset$. Note that this definition is symmetric. Now, $H \prec G$ if and only if there exists a family of disjoint sets $\{A_v : v \in H\}$ such that for every $uv \in E(H)$ the sets A_u and A_v are touching. An *induced minor* of G is a graph H obtained from G by a sequence of subgraph contractions.

A famous conjecture of Hadwiger relates the notions of a minor and of chromatic number. It states that if a graph G has no K_n -minor then $\chi(G) < n$. We will use the following result of Kostochka[3], which is the strongest known upper bound for the chromatic number in terms of a forbidden clique minor:

Theorem 2.1 ([3]). *If G is a graph with no K_n -minor then*

$$\chi(G) = O(n\sqrt{\log n})$$

A *tree* is a connected graph with no cycles. A *spanning tree* in a (multi)graph G is a tree T which is a subgraph of G such that $V(T) = V(G)$. A *rooted tree* is a tree with one selected vertex—the *root*. In a rooted tree the neighbour of a vertex v in the direction towards the root is called the *parent* of v , while all other neighbours of v are called *children* of v . An *ancestor* of a vertex v in a rooted tree is any vertex on the tree-path connecting v with the root. A rooted spanning tree T in G is *normal* if every edge in $E(G) \setminus E(T)$ connects a vertex to its ancestor. Note that running the depth-first search (DFS) algorithm for a graph G builds a normal spanning tree of G .

For a partition P of the vertices of a graph G a *crossing edge* is an edge connecting different sets of the partition. The following Nash-Williams theorem [4] will be an important tool used at the end of our main proof.

Theorem 2.2 ([4]). *A multigraph G contains k edge-disjoint spanning trees if and only if every partition P of its vertices has at least $k(|P| - 1)$ crossing edges.*

A *weighted graph* is a graph G equipped with a weight function $w : V(G) \rightarrow [0, \infty)$. We say that G is $\{0,1\}$ -*weighted* if $w : V(G) \rightarrow \{0,1\}$. For a $\{0,1\}$ -weighted graph with vertex set V we denote by V_1 the set of vertices of weight 1 and by V_0 the set of vertices of weight 0.

Observation 2.3. *Let T be a weighted tree. Then there is a vertex $v \in V(T)$ such that every component C of $T \setminus \{v\}$ has weight $w(C) \leq \frac{1}{2}w(T)$. We call v a weighted center of T .*

Proof. Think of every edge as directed towards the heavier component of $T - e$. Then there is a vertex such that all its incident edges are directed inwards. Such a vertex is a good candidate for a weighted center of T . $\square$

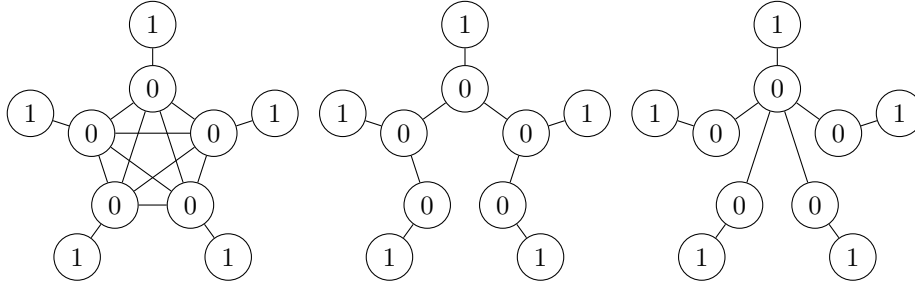
Theorem 2.4 ([6]). *Let G be a connected weighted graph with an odd number of vertices and H be a connected subgraph induced by the vertices taken at some point of the game. Then the first player has a strategy in game T to capture at least $\frac{1}{2}$ of the weight of all connected components of $G \setminus H$ except the heaviest one.*

A subset B of vertices of a graph G is *sparse* if the distance between any two vertices in B is at least 3. A $\{0,1\}$ -weighted graph G is *sparsely weighted* if $V_1(G)$ is sparse.

3 State of the art

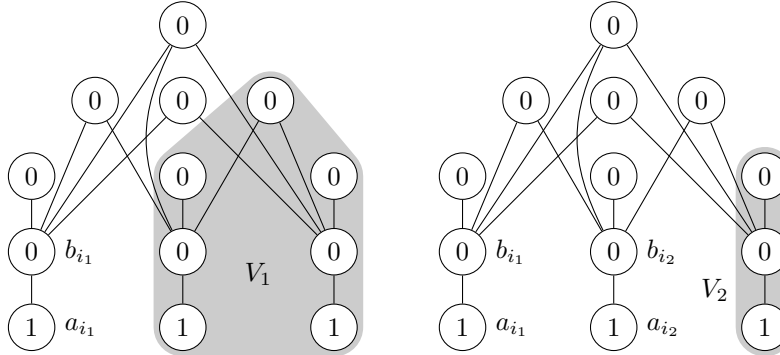
3.1 Game T

In the variant with taken part connected all vertices are available for the first player at the beginning, so obviously he can secure at least one heavy vertex. The following example shows that even on trees he cannot be sure to get more.

Figure 1: G_5 and related examples

Example 3.1. $G_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \dots, a_n, b_1, \dots, b_n\}$. The a_i 's forms a clique and the only neighbour for each b_i is a_i . The weight is equally distributed only on b_i 's, so $w(b_i) = 1$ and $w(a_1) = 0$ for every i . There is no strategy for the first player in game T to secure more than 1 in G_n . When he starts with some b_i , the second player takes the corresponding a_i . In next moves the first player is forced to take only b_i 's and second player can always take the corresponding a_i 's.

Note that the clique on a_i 's can be replaced by any other connected graph and the argument still works. This way we obtain similar examples belonging to simple classes of graphs (trees, caterpillars, subdivided stars, etc.). However, all such examples have an even number of vertices. If we request the graph to have an odd number of vertices, things get a bit more complicated, but it is still possible to construct an example for which the first player has no strategy to take more than one heavy vertex. The following example was found independently by two groups of researchers [6, 9].

Figure 2: H_3 : The first player is forced to enter sets V_1 and V_2 .

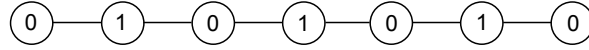
Example 3.2. $H_n = (V, E, w)$ is a weighted graph with $2n + 2^n - 1$ vertices $V = \{a_1, \dots, a_n, b_1, \dots, b_n\} \cup \{c_X : X \in P(\{b_1, \dots, b_n\}) - \{\emptyset\}\}$. The neighbourhoods

of the vertices are: $N(a_i) = \{b_i\}$, $N(b_i) = \{a_i\} \cup \{c_X : b_i \in X\}$, $N(c_X) = X$. The weight is equally distributed on the a_i 's, so $w(a_i) = 1$ for all i 's. The first player can secure at most 1 in game T on H_n . Suppose that he starts with a_{i_1} (or b_{i_1}). Then the second player takes the corresponding vertex b_{i_1} (or a_{i_1}). The set of vertices c_X which are available to play depends only on the vertices b_i that are already taken. For a non-empty set $X \subseteq \{b_1, \dots, b_n\}$ denote by V_X the set of these c_Y 's that are neighbours of some $b_i \in X$. We have $|V_X| = (2^n - 1) - (2^{n-|X|} - 1) = 2^n - 2^{n-|X|}$ even. It is still even after adding couples of a_i 's and b_i 's which are already taken. Thus the second player can ensure that he is not the one who takes a new b_i , and he can take a_i just after the first player takes b_i .

Note that $K_n \prec H_n$, so H_n is not a counterexample for Conjecture 1.1, but its chromatic and clique numbers are both equal to 2.

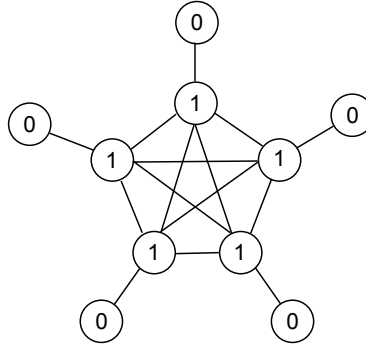
3.2 Game R

In the variant with remaining part connected the set of vertices available for the first player in the first move is restricted, so we can hope for an example leaving the first player with no weight. To see that this is possible it is enough to consider a path with vertices weighted alternately with 0 and 1.



Example 3.3. $P_n = (V, E, w)$ is a weighted path with $2n - 1$ vertices. The weight is equally distributed on every second vertex of the path. It is clear that the second player can guarantee himself all the weight.

There is also an example with an even number of vertices and with a slightly more complicated structure involving K_n as a subgraph. It is a slight modification of Example 3.1.



Example 3.4. $G'_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \dots, a_n, b_1, \dots, b_n\}$. The a_i 's form a clique and the only neighbour for each b_i is a_i . The weight is equally distributed only on the a_i 's, so $w(a_i) = 1$ and $w(b_i) = 0$ for every i . It should be clear that the second player take all heavy vertices except one, always taking the a_i corresponding to the b_i taken by the first player.

4 Proof of the main result

In this section we consider only $\{0,1\}$ -weighted graphs and game T. A *heavy* vertex is a vertex of weight 1. In the proof we first take care about heavy vertices which are close to each other, and then in a sparsely weighted graph we introduce the core definition of surrounding (we need minimal distance between two heavy vertices to be at least 3 to have the desired properties of this notion). For a sparsely weighted graph G and $B \subseteq V_1$ the *surrounding* of B is the set of all vertices $x \in V(G)$ for which there is a path $v-x$ not crossing the neighbourhood of any $v \in V_1 \setminus B$. We denote the surrounding of B by $s(B)$. Instead of $s(\{v\})$ we write just $s(v)$ for $v \in V(G)$. For a graph G the *graph of surroundings* $S(G)$ is the graph with vertex set $V(S(G)) = V_1$ such that $uv \in E(S(G))$ if $s(u)$ and $s(v)$ are touching. Note that $uv \in E(S(G))$ if and only if there is a path between u and v avoiding the neighbourhoods of other vertices in V_1 . The idea behind this definition is the following: if at some point of the game only vertices from $S(B)$ are taken and one of the players takes something from outside $S(B)$ then his opponent can take a heavy vertex in the following move.

Lemma 4.1. *Let G be a sparsely weighted graph with an odd number of vertices and $A \subseteq V_1$ be connected in $S(G)$. Then the first player has a strategy to capture at least $1/2$ of the weight of all connected components of $S(G) \setminus A$ except the heaviest one.*

Proof. The first player begins inside $s(A)$. If any heavy vertex is available then the first player takes it, otherwise he takes an arbitrary vertex from $s(A)$. When there are no more vertices from $s(A)$, he applies the strategy from Lemma 2.4.

Note that the second player cannot gather any weight until $s(A)$ is entirely taken, because G is sparsely weighted. Thus the first player captures at least $1/2$ of the weight of all connected components of $S(G) \setminus s(A)$ except the heaviest one. $\square$

Lemma 4.2. *Let G be a $\{0,1\}$ -weighted graph with an odd number of vertices such that $K_n \not\leq G$. Then one of the following holds:*

- (1) *The first player can secure $\Omega(\frac{1}{n^3 \log n})$ of the total weight of G .*
- (2) *There is a sparse set $V'_1 \subset V_1$ with $w(V'_1) = \Omega(\frac{1}{n\sqrt{\log n}^{3/2}})w(V_1)$.*

Proof. By Theorem 2.1 the vertices of G can be coloured with $O(n\sqrt{\log n})$ colours. One of the colour classes has total weight at least $\Omega(\frac{1}{n\sqrt{\log n}})w(V_1)$. The first player can play as if all vertices of other colours had weight 0. Thus we clear the weights on the vertices from all other colour classes, leaving $\Omega(\frac{1}{n\sqrt{\log n}})$ fraction of the original weight of G . The resulting graph has distance at least 2 between any two heavy vertices. Now we colour the graph again, but this time we want to avoid the same colour on two heavy vertices sharing a neighbour from V_0 which is not a neighbour of any other heavy vertex. To this end, in every such triple of vertices we contract one vertex from V_1 with the middle one from V_0 and use again the colouring by Theorem 2.1. We again clear the weights on the vertices of other colours, leaving $\Omega(\frac{1}{n^2 \log n})$ fraction of the original weight of G .

Now let B be the set of vertices from V_0 which have at least 3 heavy neighbours. A *stack* is a maximal set of vertices from $V_1 \cup B$ connected in G . A *core* of a stack S is a set $S' \subseteq S$ such that:

- (i) $N_G(S') \cap S \cap V_1 = \emptyset$.
- (ii) There is no vertex in $N_G(S')$ with more than one neighbour from $V_1 \setminus S'$.
- (iii) There is no vertex in $S \setminus S'$ with more than two neighbours from $V_1 \setminus S'$.
- (iv) All vertices from $V_1 \cap S$ are at distance at most 2 from S' .

Condition (iv) can be derived from (i)–(iii), but it is clearer to state it explicitly. The core is *light* if its weight is less than half of the weight of the stack. A stack is *small* if it has less than 9 heavy vertices, otherwise it is *big*. Depending on the structure of stacks we proceed in one of the following ways:

CASE 1. More than $\frac{1}{3}$ of the total weight of the graph is outside big stacks. Then we clear the weights in all stacks except on one vertex per each small stack. We thus obtain a sparsely weighted graph with $\Omega(\frac{1}{n^2 \log n})$ fraction of the original weight of G .

CASE 2. More than $\frac{1}{3}$ of the total weight of the graph is in big stacks with light cores. Then for each such stack S we clear all the weight from its core S' and all the weight outside all such stacks, losing at most $\frac{5}{6}$ of the weight we had before. Note that by (iii) there is no more vertex with more than two heavy neighbours, so we can use the colouring by Theorem 2.1 again to get a sparsely weighted graph with $\Omega(\frac{1}{n^3 (\log n)^{3/2}})$ fraction of the original weight of G .

CASE 3. More than $\frac{1}{3}$ of the total weight of the graph is in big stacks

without light cores. Then we clear all the weight outside those stacks and apply the following strategy for the first player:

- (1) If every stack containing a vertex available to play has a core that is entirely taken then make an arbitrary move.
- (2) Otherwise there is a stack S with no core that is entirely taken and with a vertex $v \in S$ available to play. Then at least one of the following holds:
 - (i) There is an available heavy vertex. Then simply take it.
 - (ii) There is an available vertex v with at least two not taken heavy neighbours. Then take it, and in the next move take one of its neighbours.
 - (iii) Cases (i) and (ii) do not hold. Then there is a vertex in $B \cap S$ at distance greater than 1 from the taken part (because no core have been taken yet). Let v be a closest such vertex. Note that it is at distance at most 3 because two heavy vertices cannot be adjacent. Moreover, v has at least 3 heavy neighbours which are not taken yet, because otherwise it would be adjacent to the taken part. Note that if v is taken by any of the players before two of its heavy neighbours then in the following move the first player can take at least one of them. Let P be a shortest path from the taken part to v . The first player takes the first vertex on the path. If the second player takes the second vertex on the path then the first one can take v and if the second one takes v then the first one can take a heavy vertex. Suppose then that the second player takes some other vertex. Then the first player takes the second vertex on the path making v available. In the following move the first player takes v (unless already taken by the second one), and afterwards he will take a heavy vertex.

In any of the above the first player takes a heavy vertex in less than 8 moves. Thus he can secure at least $\frac{1}{8}$ of the weight in big stacks without light cores, which is at least $\frac{1}{48}$ of the starting weight in case 3—he plays in each stack until some of its cores is taken and thus until at least half of its weight is taken. $\square$

Lemma 4.3. *Let G be a sparsely $\{0, 1\}$ -weighted graph with an odd number of vertices such that $K_n \not\prec G$. Let $A \subseteq V_1(G)$ be such that $|A| \geq n$ and $|s(A)|$ is odd. Then there is $B \subsetneq A$ such that $|B| < n$ and $|s(B)|$ is odd.*

Proof. Let $A = \{a_1, \dots, a_m\}$ and suppose that for every $B \subsetneq A$ we have $|s(B)|$ even. Pick up arbitrary $u, v \in A$. From the fact that the sum of surroundings is the surrounding of the sum we have $|s(u) \cup s(v)|$ even, same as $|s(u)|$ and $|s(v)|$, so $|s(u) \cap s(v)| = |s(u)| + |s(v)| - |s(u) \cup s(v)|$ has to be even as well. The same reasoning shows that $|s(u) \cap s(A \setminus \{u, v\})|$ and $|s(v) \cap s(A \setminus \{u, v\})|$ are even. Since $s(A) = s(u) \cup s(v) \cup s(A \setminus \{u, v\})$, by the inclusion-exclusion principle we have:

$$|s(A)| = |s(u)| + |s(v)| + |s(A \setminus \{u, v\})| - |s(u) \cap s(v)| - |s(u) \cap s(A \setminus \{u, v\})| - |s(v) \cap s(A \setminus \{u, v\})| + |s(u) \cap s(v) \cap s(A \setminus \{u, v\})|$$

Note that the left-hand side is odd. On the right-hand side $|s(u)|$, $|s(v)|$ and $|s(A \setminus \{u, v\})|$ are even, same as $|s(u) \cap s(v)|$, $|s(u) \cap s(A \setminus \{u, v\})|$ and $|s(v) \cap s(A \setminus \{u, v\})|$, so $|s(u) \cap s(v) \cap s(A \setminus \{u, v\})|$ has to be odd. Now we have

$$|(s(v) \setminus s(A \setminus \{u, v\})) \cap (s(u) \setminus s(A \setminus \{u, v\}))| = |(s(u) \cap s(v)) \setminus s(A \setminus \{u, v\})| = |(s(u) \cap s(v)) \setminus (s(u) \cap s(v) \cap s(A \setminus \{u, v\}))|$$

The right-hand side is odd, so $(s(v) \setminus s(A \setminus \{u, v\})) \cap (s(u) \setminus s(A \setminus \{u, v\})) \neq \emptyset$. Moreover, $s(A \setminus \{u\}) = s(A \setminus \{u, v\}) \cup s(v)$, which implies that $s(v) \setminus s(A \setminus \{u, v\})$ and $s(u) \setminus s(A \setminus \{u\}) \neq \emptyset$ are touching.

Let $M_1 = s(a_1)$ and $M_k = s(a_k) \setminus \bigcup_{1 \leq i < k} M_i$ for $k = 2, \dots, m$. For any i and j such that $1 \leq i < j \leq m$ we have $s(m_i) \setminus s(A \setminus \{m_i, m_j\}) \subseteq M_i$ and $s(m_j) \setminus s(A \setminus \{m_j\}) \subseteq M_j$, so by the above M_i and M_j are touching. Therefore $K_m \prec G$, which contradicts the assumption. Thus there is a set $B \subsetneq A$ with an odd number of vertices in its surrounding. We repeat the same argument until we get $|B| < n$. $\square$

Lemma 4.4. *Let G be a sparsely $\{0, 1\}$ -weighted graph with an odd number of vertices such that $K_n \not\prec G$ and let $B \subseteq V_1$ be connected in $S(G)$. Then the first player can secure at least $\frac{1}{n}w(N_{S(G)}(B))$.*

Proof. Without loss of generality we can assume that vertices outside $N_{S(G)}^+(B)$ have weight 0. Let $B_0 = B$. If $|s(B)|$ is odd then the first player keeps playing vertices from $s(B)$ forcing the second one to take a vertex from outside $s(B)$ and then takes an available heavy vertex. If $|s(B_0)|$ is even then examine the graph $G_1 = G \setminus s(B_0)$. Since G_1 has an odd number of vertices, by Lemma 4.3 there is $B'_0 \in V_1(G_1)$ with $|B'_0| < n$ such that $s_{G_1}(B'_0)$ is connected and has an odd number of vertices. Thus $s(B_0 \cup B'_0)$ also has an odd number of vertices, so the first player can force the second one to take a vertex from outside $s(B_0 \cup B'_0)$ and then takes an available heavy vertex v . Note that $|B'_0 \cup \{v\}| \leq n$, so we can set $B_1 = B_0 \cup B'_0 \cup \{v\}$ and repeat the same argument until the entire $N_{S(G)}(B)$ is taken. Using this strategy the first player can capture at least $\frac{1}{n}w(N_{S(G)}(B))$, because he takes at least one heavy vertex per at most $n - 1$ heavy vertices taken by the other player, and he can take one heavy vertex from B at the beginning. $\square$

Lemma 4.5. *Let $H = (V, E)$ be a connected graph with m vertices of degree 2. Then there is a set $A \subseteq V$ such that $|N_H(A)| \geq \frac{1}{11}m$.*

Proof. Let $A \subseteq V$ be a connected set maximal with respect to:

- the weight of $N(A)$,
- the number of vertices.

It follows from the definition that $N(A) = V$. Indeed, suppose there is a vertex $v \in V$ at distance 2 from A . Then we can add a common neighbour of A and v to A and make A bigger with at least the same number of neighbours. Let $B = N(A)$. Suppose there is $v \in A$ which is not a cut point of $G[A]$ and which

does not have a private neighbour in B (that is, a neighbour in B which is not adjacent to any other vertex in A). Then we can remove it from A making its neighbourhood bigger. So it follows that every vertex v in A is a cut vertex of A or has a private neighbour in B . Let E denote the set of vertices in A with a private neighbour in B . Clearly $|E| \leq |B|$. Note that all vertices of degree 1 in T are in E since they cannot be cut vertices.

Now, let T be a rooted normal spanning tree of $G[A]$. The following kinds of vertices in H may have degree other than 2:

- (i) Vertices from B .
- (ii) Branching vertices of T (that is, vertices of degree greater than 2 in T). The number of branching vertices is less than the number of leaves, and since leaves are in E , there are at most $|B|$ of them.
- (iii) Upper ends of jumping edges in T . Each such vertex is a unique parent of a branching vertex or of a vertex from E . Thus there are at most $2|B|$ vertices of this kind.
- (iv) Lower ends of jumping edges in T . Each such vertex is a child of a branching vertex or a unique child of a vertex from E that has degree 2 in T . The total number of children of branching vertices is less than twice the number of leaves. Thus the number of vertices of this type is at most $3|B|$.
- (v) Vertices that have a neighbour in B . Suppose there are at least $3|B|$ such vertices. By the following inductive procedure we construct a connected set C with more neighbours than $|N(A)|$:

- Start with $C = B$.
- Take any vertex $b \in B$ and look at its connected component in C . Find a shortest path from this component to any vertex $b' \in B$ outside this component and add the entire path to C . Repeat until C is connected.

In a single application of the second step we add to C at most two vertices from $N(B)$ and we need at most $|B| - 1$ such steps to make C connected. Thus more than $|B|$ vertices from $N(B)$ remain outside C at the end and all of them are in $N(C)$.

Summing up all the bounds we get less than $10|B|$ vertices in H of degree other than 2. Since m vertices in H have degree 2, there is a connected set A whose neighbourhood has size at least $\frac{1}{11}m$. $\square$

Lemma 4.6. *Let G be a sparsely $\{0,1\}$ -weighted graph with an odd number of vertices such that $K_n \not\prec G$. Then at least one of the following holds:*

- (1) *The first player can secure at least $\frac{1}{22n}w(G)$ in game T .*
- (2) *At least $\frac{1}{2}$ of heavy vertices have degree 2 in $S(G)$.*

Proof. Suppose that less than $\frac{1}{2}$ of heavy vertices have degree 2 in $S(G)$. Then by Lemma 4.5 there is a set $B \subseteq V(S(G))$ such that $|N_{S(G)}(B)| \geq \frac{1}{22}|V_1|$, so by Lemma 4.4 the first player can capture at least $\frac{1}{22n}w(G)$. $\square$

Lemma 4.7 ([7]). *Let G be a sparsely $\{0,1\}$ -weighted graph with an odd number of vertices such that $S(G)$ is a cycle. Then the first player can secure at least $\frac{1}{6}$ of the total weight of G .*

Lemma 4.8. *Let G be a $\{0,1\}$ -weighted graph such that all vertices of weight 1 have degree 2. Then one of the following holds:*

- (1) *There is a connected set $Z \subseteq V(G)$ such that the total weight of all connected components of $G \setminus Z$ except any one of them is at least $\Theta(w(G))$.*
- (2) *G has an induced minor which is a cycle and has weight at least $\Theta(w(G))$ on non-contracted vertices.*

Proof. Let T be a normal spanning tree of G and $v \in V(G)$ be its weighted center. Let u be the highest ancestor of v in T adjacent to a vertex w which is below v in T . Let C be the cycle obtained from joining the tree-paths $u - v$, $v - w$ and the edge wu . We distinguish two cases:

CASE 1. $w(C) < \frac{1}{4}w(G)$. Note that in $G \setminus C$ the parts on which v partitions T becomes disconnected, because T is a normal spanning tree and no more edge jumping over v remains. None of these parts is heavier than $\frac{1}{2}w(G)$, so the total weight of all connected components of $G \setminus C$ except the heaviest one is at least $\frac{1}{2}w(G) - w(C) > \frac{1}{4}w(G)$.

CASE 2. $w(C) \geq \frac{1}{4}w(G)$. Clear the weights on all vertices outside C . Contract each connected component of $G \setminus C$ with an arbitrarily chosen vertex in C adjacent to that component. Let H be the graph obtained after all contractions. Note that all contracted vertices of G have weight 0, so $w(H) \geq \frac{1}{4}w(G)$ and the vertices of H with weight 1 have degree 2.

Now it suffices to prove the statement of the lemma for the graph H . Indeed, if (1) holds for H and a set $A \subseteq V(H)$ then the set of vertices from G that contract to A satisfies (1) for G . And if (2) holds for H and an induced minor B then B is also an appropriate induced minor of G , since all vertices of H with weight 1 result from single vertices of G .

For the remainder of the proof we assume that G is a $\{0,1\}$ -weighted Hamiltonian graph (that is, a graph with a cycle passing through each vertex exactly once) with weights 1 on vertices of degree 2 only.

We now describe a procedure that aims at finding in G an appropriate set Z satisfying (1) or an induced cycle-minor satisfying (2). The procedure maintains a weighted Hamiltonian graph H and a set $A \subseteq V(H)$. Denote by C the Hamiltonian cycle in H and by $A_1, \dots, A_k$ the connected components of $H[A]$. Define the *component multigraph* M with vertices $A_1, \dots, A_k$ so that there is one edge between A_i and A_j ($i \neq j$) per each path on C connecting a vertex from A_i with a vertex from A_j and not passing through any other vertex from A . Note that the degree of each vertex of M is even. Define the *region* $R(A_i)$ of a component A_i to be the set A_i together with all vertices in paths on C connecting A_i to itself and not passing through any other vertex from A . The procedure maintains the following invariants:

- (i) H is an induced minor of G . Each vertex v of H either corresponds to a single vertex of G or results from contracting a connected segment $S(v)$ of

the Hamiltonian cycle of G . We call the vertices of the latter kind *super-vertices*. The weight of a super-vertex v is the total weight of the vertices in $S(v)$.

- (ii) Each super-vertex has degree 2.
- (iii) For super-vertex v we have $w(S(v)) \leq \frac{1}{8}w(G)$.
- (iv) For each super-vertex v there is a set $T(v) \subseteq S(v)$ of vertices in G connecting the first and the last vertex of $S(v)$ and such that $w(S(v)) \geq 2w(T(v))$.
- (v) Each connected component A_i of $H[A]$ satisfies $w(R(A_i)) \geq 2w(A_i)$.

We start with $H = G$ and A being the set of vertices of degree greater than 2 in G . There are no super-vertices and all vertices in A have weight 0, so the invariants are clearly satisfied. In a single step in the procedure we distinguish several cases.

CASE 1. All vertices of M have degree at least 4. Let P be a finest partition of the vertices of M which has less than $2(|P| - 1)$ crossing edges. Note that since the vertices have degree at least 4, the trivial partition into singletons does not satisfy this condition. Therefore, there is $X \in P$ with $|X| \geq 2$. The induced submultigraph $M[X]$ satisfies the assumptions of Theorem 2.2 for $k = 2$ (otherwise there would be a partition P' of $M[X]$ with less than $2(|P'| - 1)$ crossing edges and we could refine P replacing X by P'). It follows that $M[X]$ has two edge-disjoint spanning trees. Each of them corresponds to a set of paths in C connecting elements of X regarded as connected components of $H[A]$. We take the tree for which the total weight of all vertices on these paths is smaller and add all those vertices to the set A . This way all components of $H[A]$ from X become connected into one single component. The region of the new component contains the regions of the components in X as well as all vertices on the paths in C corresponding to the edges of the other spanning tree. Since these vertices are heavier than those added to A , the invariant (v) is maintained.

CASE 2. There is a vertex A_i in M of degree 2. Note that the region of A_i is a connected segment of C , the first and the last vertex of which belong to A_i . Here we distinguish three subcases:

CASE 2.1. $w(R(A_i)) \leq \frac{1}{8}w(G)$. Then we contract $R(A_i)$ to a super-vertex in H and remove A_i from A . Invariants (i)–(iii) and (v) are clearly satisfied and invariant (iv) follows directly from invariant (v) applied to A_i .

CASE 2.2. $\frac{1}{8}w(G) < w(R(A_i)) \leq \frac{3}{4}w(G)$. Then we argue that statement (1) of the theorem holds for $Z = A_i$. Indeed, $V(H) \setminus R(A_i)$ and $R(A_i) \setminus A_i$ are disconnected in $H \setminus A_i$, and moreover $w(V(H) \setminus R(A_i)) \geq \frac{1}{4}w(G)$ and $w(R(A_i) \setminus A_i) > \frac{1}{16}w(G)$ by invariant (v).

CASE 2.3. $w(R(A_i)) > \frac{3}{4}w(G)$. Then $w(R(A_i) \setminus A_i) > \frac{3}{8}w(G)$ by invariant (v). Let D be the set of super-vertices in $R(A_i) \setminus A_i$. Suppose first that $w(D) \geq \frac{2}{8}w(G)$. Then the total weight of all super-vertices in D except any one of them is at least $\frac{1}{8}w(G)$. Now we set $Z = (R(A_i) \setminus D) \cup \{T(v) : v \in D\}$. The sets $S(v) \setminus T(v)$ taken for all super-vertices $v \in D$ are disconnected in $G \setminus Z$, and by invariant (iv) the total weight of sets $S(v) \setminus T(v)$ taken for all super-vertices $v \in D$ except any one of them is at least $\frac{1}{16}w(G)$. Thus (1) holds.

Suppose now that $w(D) < \frac{2}{8}w(G)$. Then $w(R(A_i) \setminus (A_i \cup D)) \geq \frac{1}{8}w(G)$.

Connected components of $R(A_i) \setminus A_i$ in H are segments of C consisting of vertices of degree 2 in H , some of them being super-vertices. If all these segments except any one of them have weight at least $\frac{1}{16}w(G)$ then we set $Z = A_i$ and conclude that (1) holds. Otherwise one of the segments has weight at least $\frac{1}{16}w(G)$ on ordinary vertices. Then we contract each super-vertex on that segment as well as the complement of it thus obtaining an induced cycle-minor in G satisfying (2).

CASE 3. $A = \emptyset$. Let D be the set of super-vertices in H . Just as in Case 2.3, if D is heavy then we argue that setting $Z = (V(H) \setminus D) \cup \{T(v) : v \in D\}$ satisfies (1), otherwise we contract all super-vertices and thus find an induced cycle-minor in G satisfying (2). $\square$

Corollary 4.9. *Let G be a $\{0,1\}$ -weighted graph with an odd number of vertices in which the vertices of degree 2 carry weight at least $\frac{1}{2}w(G)$. Then the first player can collect $\Theta(w(G))$.*

Proof. Applying Lemma 4.8 to the graph $S(G)$ we obtain either an induced cycle-minor of weight $\Theta(w(G))$ or a set Z splitting $S(G)$ into balanced components. In the first case we clear the weights on contracted vertices and obtain a graph G' such that $S(G')$ is a cycle, so the conclusion follows from Lemma 4.7. In the latter case the conclusion follows by Lemma 4.1. $\square$

Proof of Theorem 1.2:

Proof. By Lemma 4.2 we obtain a sparsely weighted graph G' with $w(G') = \Omega(\frac{1}{\sqrt{n \log n}^3})w(G)$ or a strategy for the first player to gather at least as much. In the first case the desired strategy of the first player is obtained by Corollary 4.9 if most of the weight of G' lies on vertices of degree 2 or by Lemma 4.6 otherwise. $\square$

5 Summary and further directions

Although the proof is made for the $\{0,1\}$ -weighted case, it is possible to translate some of its parts into the case of general weights. Lemmas 4.1, 4.7 and Corollary 4.9 require little modification to be generalised, and there is a different proof of the Lemma 4.4 giving a worse bound but working in the general case too. The biggest problem in generalising the argument are Lemmas 4.2 and 4.6. In particular, there is a counterexample showing that the present statement of Lemma 4.6 is not true for general weights. We hope that it will be possible to devise a similar argument where a modification is needed only in the case where Lemma 4.6 played role using Lemma 4.1 again.

There are more problems in the area, the most intriguing ones arising when we consider graphs with an odd number of vertices in game T and an even number of vertices in game R . It looks to be an extremely hard problem to find the exact constant that the first player can secure under certain conditions

(minor-free etc.). Easier but still very interesting are problems concerning the necessary conditions for a class of weighted graphs to ensure that the first player has a strategy to secure a linear fraction of the total weight of any graph in this class.

An instance of such question for game T is Conjecture 1.1, and it seems that forbidden minors are the weakest natural graph property that can guarantee a linear gain for the first player. Note that the assumption about a forbidden minor is used in only two places in the proof—Lemma 4.4 and the first part of Lemma 4.2. It seems possible to prove Lemma 4.2 without using forbidden minors. Then we can ask here what property of weighted graphs is required to be sure that the first player is able to capture linearly much weight. For example, this could be some property of $S(G)$ or some quantitative requirement on number of odd surroundings.

In game R it is not clear what parameter is responsible for the existence of a good strategy for the first player. In Example 3.4 the k -clique is indeed required so it looks natural to ask about the clique number or at least the chromatic number. Unfortunately it is not even known if the first player can secure linearly much weight on bipartite graphs.

Conjecture 5.1. *Let G be a weighted bipartite graph with an even number of vertices. Then the first player can secure $\Theta(w(G))$ in game R .*

Another direction one could head is to improve the constant which the first player can secure for more restricted classes of graphs like planars or outer-planars. They are both characterized by forbidden minors, but they can be also equipped with an embedding together with notions of a left/right and an up/down vertex, which could be helpful in finding a strategy.

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